SHEET 8 Solutions 28/12/2020

Exercise 1 Find e-values & e-vectors for A

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 6 & 4 \\ 0 & 3 & 2 \end{pmatrix}$$

▶ Charac. Polynomial $p(\lambda) = |\lambda I - A|$

$$\left| \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 \\ 0 & 6 & 4 \\ 0 & 3 & 2 \end{pmatrix} \right| = \begin{vmatrix} (\lambda-1) & -2 & -1 \\ 0 & (\lambda-6) & -4 \\ 0 & -3 & (\lambda-2) \end{vmatrix}$$

$$= (\lambda-1)\left[(\lambda-6)(\lambda-2) - 12\right] - (-2)\left[0-0\right] + (-1)\left[0-0\right]$$

$$= (\lambda-1)\left[\lambda^2 - 2\lambda - 6\lambda + 12 - 12\right] + 0$$

$$p(\lambda) = (\lambda-1)(\lambda^2 - 8\lambda) \longrightarrow \lambda^3 - 9\lambda^2 + 8\lambda \quad (1)$$

▶ $0 = (\lambda-1)\lambda(\lambda-8) \longrightarrow$ $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 8$ (eigenvalues)

▶ Eigenvectors

• $\lambda_1 = 0$ (1)

$(\lambda I - A)\underline{x} = 0$

$$\begin{pmatrix} (\lambda-1) & -2 & -1 \\ 0 & (\lambda-6) & -4 \\ 0 & -3 & (\lambda-2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -2 & -1 \\ 0 & -6 & -4 \\ 0 & -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Row Reduce

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -6 & -4 \\ 0 & -3 & -2 \end{pmatrix} \xrightarrow[-R_1]{R_3 - \frac{1}{2}R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -6 & -4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{6}R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{pmatrix}$$

System of equations

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 0 + x_2 + \frac{2}{3}x_3 = 0 \end{cases}$$

set $x_3 = t$

$x_2 = -\frac{2}{3}t$

$x_1 = -\frac{2}{3}t + t = +\frac{1}{3}t$

$$\underline{v}_1 = \begin{pmatrix} +\frac{1}{3}t \\ -\frac{2}{3}t \\ t \end{pmatrix} \rightsquigarrow \begin{pmatrix} +1 \\ -2 \\ 3 \end{pmatrix} t \quad \text{b/c multiples}$$

$$\begin{pmatrix} 1/3 \\ -2/3 \\ 1 \end{pmatrix} t$$

• $\lambda_2 = 1$

(1)

$$\begin{pmatrix} 0 & -2 & -1 \\ 0 & -5 & -4 \\ 0 & -3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

System of equations

Row Reduce

$$\xrightarrow[R_3 - (3/2)R_1]{R_2 - (5/2)R_1} \begin{pmatrix} 0 & -2 & -1 \\ 0 & 0 & 3/2 \\ 0 & 0 & -1/2 \end{pmatrix} \xrightarrow[R_3 + (1/3)R_2]{-R_1/2} \begin{pmatrix} 0 & 1 & 1/2 \\ 0 & 0 & 3/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[(2/3)R_2]{R_1 - (1/3)R_2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

System of equations

$$\begin{cases} 0 + x_2 + 0 = 0 \\ 0 + 0 + x_3 = 0 \\ 0 + 0 + 0 = 0 \end{cases} \qquad x_3 = t \qquad x_2 = 0 \qquad x_1 = t$$

$$\underline{v}_2 = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t$$

• $\lambda_3 = 8$

(1)

$$\begin{pmatrix} 7 & -2 & -1 \\ 0 & 2 & -4 \\ 0 & -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Row Reduce

$$\xrightarrow[1/7\ R_1]{R_3 + (3/2)R_2} \begin{pmatrix} 1 & -2/7 & -1/7 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[1/2\ R_2]{R_1 + (1/7)R_2} \begin{pmatrix} 1 & 0 & -5/7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

System of equations

$$\begin{cases} x_1 + 0 - (5/7)x_3 = 0 \\ 0 + x_2 - 2x_3 = 0 \\ 0 + 0 + 0 = 0 \end{cases} \qquad x_3 = t \qquad x_2 = 2t \qquad x_1 = (5/7)t$$

$$\underline{v}_3 = \begin{pmatrix} 5/7\ t \\ 2t \\ t \end{pmatrix} = \begin{pmatrix} 5/7 \\ 2 \\ 1 \end{pmatrix} t$$

Exercise 2 Find P & D. Check $PD = AP$ $\{D = P^{-1}AP\}$

- from eigenvectors,

$$P = \begin{pmatrix} 1/3 & 1 & 5/7 \\ -2/3 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix} \quad (1)$$

- from eigenvalues,

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix} \quad (1)$$

- Check $PD = AP$

$$PD = \begin{pmatrix} 1/3 & 1 & 5/7 \\ -2/3 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix} = \begin{pmatrix} (0+0+0) & (0+1+0) & (0+0+40/7) \\ (0+0+0) & (0+0+0) & (0+0+16) \\ (0+0+0) & (0+0+0) & (0+0+8) \end{pmatrix}$$

$$(1) = \begin{pmatrix} 0 & 1 & 40/7 \\ 0 & 0 & 16 \\ 0 & 0 & 8 \end{pmatrix} \checkmark$$

$$AP = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 6 & 4 \\ 0 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1/3 & 1 & 5/7 \\ -2/3 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} (1/3 - 4/3 + 1) & (1+0+0) & (5/7 + 4 + 1) \\ (0 - 4 + 4) & (0+0+0) & (0 + 12 + 4) \\ (0 + 2 - 2 + 2) & (0+0+0) & (0+6+2) \end{pmatrix}$$

$$(1) = \begin{pmatrix} 0 & 1 & 40/7 \\ 0 & 0 & 16 \\ 0 & 0 & 8 \end{pmatrix} \checkmark$$

same :)

$\therefore PD = AP$